Proofs for 'Mass Splitting for Jitter-Free Parallel Rigid Body Simulation'

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1 Introduction

This document supplements the SIGGRAPH paper, Mass Splitting for Jitter-Free Parallel Rigid Body Simulation [Tonge et al. 2012], which defines the variables and notations used here.

2 Convergence Proof for Scalar Mass Splitting Algorithm

Lemma: For $\mathbf{v}^i \in \mathbb{R}^m, 1 \le i \le n$,

$$\left(\sum_{i=1}^{n} \mathbf{v}^i\right) \cdot \left(\sum_{i=1}^{n} \mathbf{v}^i\right) \le n \sum_{i=1}^{n} \mathbf{v}^i.\mathbf{v}^i \tag{1}$$

Proof:

First we prove for all $x_i \in \mathbb{R}$,

$$\left(\sum_{i=1}^{n} x_i\right)^2 \le n \sum_{i=1}^{n} x_i^2 \tag{2}$$

Proof is by induction on n, using the inequality $2xy < x^2 + y^2$. Now apply Equation 2 to every element of $\mathbf{v}_i$ in Equation 1.

Theorem:

The following iteration converges,

$$\mathbf{z}_{r+1} = (\mathbf{z}_r - \mathbf{E}(\mathbf{q} + \mathbf{N}\mathbf{z}^r))^+$$
$$\mathbf{E}_{ii} = \left(\sum_{j=1}^{n} \mathbf{J}_{ij}(n_j \mathbf{M}_j^{-1})\mathbf{J}_{ij}^T\right)^{-1}. \tag{3}$$

Proof:

We will rely on [Murty 1988], equation 9.10. For each body i, its inverse mass matrix $\mathbf{M}_i^{-1}$ is positive definite, and for $j = 1,2$ $n_{b_{i,j}} > 0$, $\mathbf{J}_{i,j}^T$ is non-zero, hence each element of the diagonal matrix $\mathbf{E}^{-1}$ is positive. This satisfies the first condition of Eq. 9.8 [Murty 1988]. The second condition applied to iteration (3) simplifies to $2\mathbf{E}^{-1} - \mathbf{N} > 0$. Since $\mathbf{E}^{-1}$ is positive-definite, it is sufficient to prove that $\mathbf{E}^{-1} - \mathbf{N}$ is positive semi-definite. Since for each body i $\mathbf{M}_i^{-1}$ is positive definite, it is possible to factor it into $\mathbf{M}_i^{-1} = \mathbf{Q}_i\mathbf{Q}_i^T$, where $\mathbf{Q}_i$ is lower triangular. For arbitrary $\mathbf{x}$, define $\mathbf{v}$ as follows,

$$\mathbf{v}^{i,j} = \mathbf{Q}_{b_{i,j}} \mathbf{J}_{i,j}^T \mathbf{x}_i. \tag{4}$$

We need the index set of the non-zero blocks of J that affect body k, defined as follows,

$$C(k) = \{(i,j) | b_{i,j} = k\}, \tag{5}$$

so $|C(k)| = n_k$. Now,

$$\mathbf{x}^T\mathbf{E}^{-1}\mathbf{x} = \sum_{i=1}^{m} \left(n_{b_{i,1}} \mathbf{v}^{i,1}.\mathbf{v}^{i,1} + n_{b_{i,2}} \mathbf{v}^{i,2}.\mathbf{v}^{i,2}\right) \tag{6}$$

$$= \sum_{k=1}^{n} \left(n_k \sum_{(i,j)\in C(k)} \mathbf{v}^{i,j}.\mathbf{v}^{i,j}\right) \tag{7}$$

$$\mathbf{x}^T\mathbf{N}\mathbf{x} = \sum_{k=1}^{n} \left(\sum_{(i,j)\in C(k)} \mathbf{v}^{i,j}\right) \cdot \left(\sum_{(i,j)\in C(k)} \mathbf{v}^{i,j}\right) \tag{8}$$

By the previous lemma, every term of $\mathbf{x}^T\mathbf{E}^{-1}\mathbf{x}$ is greater or equal to the corresponding term of $\mathbf{x}^T\mathbf{N}\mathbf{x}$ and

$$\mathbf{x}^T\mathbf{E}^{-1}\mathbf{x} \ge \mathbf{x}^T\mathbf{N}\mathbf{x}. \tag{9}$$

So, $\mathbf{x}^T(\mathbf{E}^{-1} - \mathbf{N})\mathbf{x} \ge 0$, which means that $\mathbf{E}^{-1} - \mathbf{N}$ is positive semi-definite and therefore the iteration converges.

3 Convergence Proof for Iterative MLCP with Exact Joints

Apply LCP iteration (9.9) from [Murty 1988] with $\lambda = 1$, $\omega = 1$,

$$\mathbf{E} = \begin{bmatrix} \mathbf{D}_{CC}^{-1} & 0 \\ 0 & \mathbf{N}_{FF}^{-1} \end{bmatrix}, \mathbf{K} = \begin{bmatrix} \mathbf{L}_{CC} & 0 \\ \mathbf{N}_{FC} & 0 \end{bmatrix}, \tag{10}$$

and define a new projection operator P that applies $(+)$ only to $\mathbf{z}_C$. This gives the iteration

$$\mathbf{z}_C^{r+1} = \left(\mathbf{z}_C^r - \mathbf{D}_{CC}^{-1}\left(\mathbf{q}_C + \mathbf{L}_{CC}\mathbf{z}_C^{r+1} + \begin{bmatrix} \mathbf{U}_{CC} & \mathbf{N}_{CF} \end{bmatrix} \begin{bmatrix} \mathbf{z}_C \\ \mathbf{z}_F \end{bmatrix}^r\right)\right)^+ \tag{11}$$

$$\mathbf{z}_F^{r+1} = \mathbf{z}_F^r - \mathbf{N}_{FF}^{-1}\left(\mathbf{q}_F + \begin{bmatrix} \mathbf{N}_{FC} & \mathbf{N}_{FF} \end{bmatrix} \begin{bmatrix} \mathbf{z}_C^{r+1} \\ \mathbf{z}_F^r \end{bmatrix}\right), \tag{12}$$

which has the same form as the block mass splitting iteration from the paper. To prove convergence we just need to show [Murty 1988] that for all $\mathbf{y} \ge 0$,

$$\left(\mathbf{z}^P - \mathbf{z}\right)^T \mathbf{E}^{-1} \left(\mathbf{z}^P - \mathbf{y}\right) \ge 0. \tag{13}$$

Proof:

$$\begin{aligned} &\left(\mathbf{z}^P - \mathbf{z}\right)^T \mathbf{E}^{-1} \left(\mathbf{z}^P - \mathbf{y}\right) \\ =& \begin{bmatrix} \mathbf{z}_C^+ - \mathbf{z}_C \\ \mathbf{z}_F - \mathbf{z}_F \end{bmatrix}^T \begin{bmatrix} \mathbf{D}_{CC} & 0 \\ 0 & \mathbf{N}_{FF} \end{bmatrix} \begin{bmatrix} \mathbf{z}_C^+ - \mathbf{y}_C \\ \mathbf{z}_F - \mathbf{y}_F \end{bmatrix} \\ =& (\mathbf{z}_C^+ - \mathbf{z}_C)^T \mathbf{D}_{CC} (\mathbf{z}_C^+ - \mathbf{y}_C) \\ \ge& 0 \text{ (by theorem 9.8 in [Murty 1988]).} \end{aligned} \tag{14}$$

4 Analytical Solution for Fixed Joints

For notational convenience, let $\mathbf{S}(n,m)$ be a matrix of size $(6n, 6m)$ whose elements are all one, let $\mathbf{I}(n,m)$ be the identity matrix of the

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same size and define $\mathbf{S}(n) = \mathbf{S}(n,n)$ and $\mathbf{I}(n) = \mathbf{I}(n,n)$. Now we can write

$$\mathbf{F}_\beta = \begin{bmatrix} \mathbf{S}(n_\beta - 1, 1) & -\mathbf{I}(n_\beta - 1) \end{bmatrix}. \tag{15}$$

The following proof will make use of the identity

$$\mathbf{S}(n,m)\mathbf{S}(m,p) = m\mathbf{S}(n,p). \tag{16}$$

Theorem:

$$\left(\mathbf{I}(n_\beta) - \mathbf{F}_\beta^T(\mathbf{F}_\beta \mathbf{W}_\beta \mathbf{F}_\beta^T)^{-1}\mathbf{F}_\beta \mathbf{W}_\beta\right) = n_\beta^{-1}\mathbf{S}(n_\beta). \tag{17}$$

Proof:

$$\begin{aligned}(\mathbf{F}_\beta \mathbf{W}_\beta \mathbf{F}_\beta^T)^{-1} &= n_\beta^{-1}\mathbf{M}_\beta \left(\mathbf{I}(n_\beta - 1) + \mathbf{S}(n_\beta - 1)\right)^{-1} \\ &= n_\beta^{-1}\mathbf{M}_\beta \left(\mathbf{I}(n_\beta - 1) - n_\beta^{-1}\mathbf{S}(n_\beta - 1)\right),\end{aligned} \tag{18}$$

and

$$\mathbf{F}_\beta^T \left(\mathbf{I}(n_\beta - 1) - n_\beta^{-1}\mathbf{S}(n_\beta - 1)\right)\mathbf{F}_\beta = \mathbf{I}(n_\beta) - n_\beta^{-1}\mathbf{S}(n_\beta). \tag{19}$$

So,

$$\begin{aligned}&\mathbf{I}(n_\beta) - \mathbf{F}_\beta^T(\mathbf{F}_\beta \mathbf{W}_\beta \mathbf{F}_\beta^T)^{-1}\mathbf{F}_\beta \mathbf{W}_\beta \\ =&\mathbf{I}(n_\beta) - n_\beta^{-1}\mathbf{M}_\beta \mathbf{F}_\beta^T \left(\mathbf{I}(n_\beta - 1) - n_\beta^{-1}\mathbf{S}(n_\beta - 1)\right)\mathbf{F}_\beta \mathbf{W}_\beta \\ =&n_\beta^{-1}\mathbf{S}(n_\beta).\end{aligned} \tag{20}$$

Corollary:

The fixed joints of body β can be solved by setting the velocity of all its sub-bodies $i \in [1..n_\beta]$ to

$$\left[\left(\mathbf{v}^S - \mathbf{W}\mathbf{F}^T(\mathbf{F}\mathbf{W}\mathbf{F}^T)^{-1}\mathbf{F}\mathbf{v}^S\right)_\beta\right]_i = n_\beta^{-1}\sum_{k=1}^{n_\beta}[\mathbf{V}_\beta]_k. \tag{21}$$

5 Proof That a Solution of the Split System is a Solution of the Original System

Theorem:

$$\begin{aligned}&\begin{bmatrix} \mathbf{z}_C \\ \mathbf{z}_F \end{bmatrix} = \mathrm{MLCP}\left(\begin{bmatrix} \mathbf{C} \\ \mathbf{F} \end{bmatrix} \mathbf{W} \begin{bmatrix} \mathbf{C} \\ \mathbf{F} \end{bmatrix}^T, \begin{bmatrix} \mathbf{q} \\ \mathbf{0} \end{bmatrix}\right) \\ \implies& \\ &\mathbf{z}_C = LCP(\mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T, \mathbf{q}).\end{aligned} \tag{22}$$

Proof:

Suppose that $\begin{bmatrix} \mathbf{z}_C \\ \mathbf{z}_F \end{bmatrix}$ is a solution of the split system, so that

$$\mathbf{C}\mathbf{v}^S + \mathbf{q} \geq 0 \perp \mathbf{z}_C \geq 0, \tag{23}$$

where $\mathbf{v}^S$ is the velocity,

$$\mathbf{v}^s = \mathbf{W}(\mathbf{C}^T\mathbf{z}_C + \mathbf{F}^T\mathbf{z}_F). \tag{24}$$

Let $e(\mathbf{v}^s)$ be the vector consisting of the velocity of the first sub-body of each body. We need to show that $\mathbf{z}_C$ is a solution of the unsplit system. We do this in two stages. First we show that if impulse $\mathbf{z}_C$ is applied to the unsplit system then the velocity of the bodies is $e(\mathbf{v}^s)$. Then we show that the velocities $e(\mathbf{v}^s)$ satisfy the constraints of the unsplit system. Applying impulse $\mathbf{z}_C$ to the unsplit system gives the following velocity for each body β,

$$\begin{aligned}&(\mathbf{M}^{-1}\mathbf{J}^T\mathbf{z}_C)_\beta \\ &= \mathbf{W}_\beta n_\beta \sum_{k=1}^{n_b}(\mathbf{C}_\beta^T\mathbf{z}_C)_k \\ &= \mathbf{W}_\beta\left(\mathbf{I} - \mathbf{F}_\beta^T(\mathbf{F}_\beta\mathbf{W}_\beta\mathbf{F}_\beta^T)^{-1}\mathbf{F}_\beta\mathbf{W}_\beta\right)\mathbf{C}_\beta^T\mathbf{z}_C \\ &= \left[\mathbf{W}(\mathbf{C}^T\mathbf{z}_C + \mathbf{F}^T\mathbf{z}_F)_\beta\right]_1.\end{aligned} \tag{25}$$

Therefore,

$$\mathbf{M}^{-1}\mathbf{J}^T\mathbf{z}_C = e(\mathbf{v}^S). \tag{26}$$

Now we show that the velocities of the split system satisfy the constraints of the unsplit system,

$$\begin{aligned}(\mathbf{C}\mathbf{v}^S)_\alpha &= \sum_{\beta=1}^{n}\mathbf{C}_{\alpha\beta}\mathbf{V}_\beta \\ &= \sum_{\beta=1}^{n}\sum_{j=1}^{n_\beta}[\mathbf{C}_{\alpha\beta}]_{1,j}\mathbf{v}_\beta \\ &= (\mathbf{J}e(\mathbf{v}^S))_\alpha.\end{aligned} \tag{27}$$

So,

$$\begin{aligned}&\mathbf{C}\mathbf{v}^S + \mathbf{q} \geq 0 \perp \mathbf{z}_C \geq 0 \\ \implies& \mathbf{J}\mathbf{M}^{-1}\mathbf{J}^T\mathbf{z}_C + \mathbf{q} \geq 0 \perp \mathbf{z}_C \geq 0.\end{aligned} \tag{28}$$

References

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